Linearization-1

Linearization: Difficult (and sometimes impossible) to obtain solutions to non-linear differential equations

* But, can examine how system behaves near equilibrium by making equations linear near equilibrium points

Scalar case: $\dot{x} = f(x,u)$

* u is some input (ex: V, f_{app}, etc.)

let $x = x_{eq} + \Delta x$ $\qquad \dot{x} = \Delta\dot{x}$

$u = \hat{u} + \Delta u$

$\Delta\dot{x} = f(x_{eq}+\Delta x, \hat{u}+\Delta u)$

Taylor series expand: $f(x_{eq}+\Delta x, \hat{u}+\Delta u) = \underbrace{\cancel{f(x_{eq}, \hat{u})}}_{0} + \left.\frac{\partial f}{\partial x}\right|_{x_{eq},\hat{u}} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_{eq},\hat{u}} \Delta u + O(\Delta x^2, \Delta u^2)$

$$f(x_{eq}+\Delta x, \hat{u}+\Delta u) \approx \left.\frac{\partial f}{\partial x}\right|_{x_{eq},\hat{u}} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_{eq},\hat{u}} \Delta u$$

$$\boxed{\Delta\dot{x} = \left(\left.\frac{\partial f}{\partial x}\right|_{x_{eq},\hat{u}}\right)\Delta x + \left(\left.\frac{\partial f}{\partial u}\right|_{x_{eq},\hat{u}}\right)\Delta u}$$

Ex $\dot{x} = -\sin(x)$

$f = -\sin(x)$ $\qquad x_{eq} = 0$

$\frac{\partial f}{\partial x} = -\cos(x)$ $\qquad \left.\frac{\partial f}{\partial x}\right|_{x_{eq}} = -1$

$\boxed{\Delta\dot{x} = -\Delta x}$

Ex $\dot{x} = -x(1+x^2)$

$f = -x(1+x^2)$ $\qquad x_{eq} = 0$

$\frac{\partial f}{\partial x} = -1(1+x^2) - 2x^2$

$\left.\frac{\partial f}{\partial x}\right|_{x_{eq}} = -1$

$\boxed{\Delta\dot{x} = -\Delta x}$

Linearization-2

Vector case:

$$\dot{x}_1 = f_1(x_1, x_2, u)$$
$$\dot{x}_2 = f_2(x_1, x_2, u)$$

$x_1 = x_{1eq} + \Delta x_1$ $\qquad u = \hat{u} + \Delta u$

$x_2 = x_{2eq} + \Delta x_2$

$$\Delta\dot{x}_1 = f_1(x_{1eq} + \Delta x_1, x_{2eq} + \Delta x_2, \hat{u} + \Delta u)$$
$$\Delta\dot{x}_2 = f_2(x_{1eq} + \Delta x_1, x_{2eq} + \Delta x_2, \hat{u} + \Delta u)$$

Taylor Series expand: $f_1(x_{1eq} + \Delta x_1, x_{2eq} + \Delta x_2, \hat{u} + \Delta u) = \cancel{f_1(x_{1eq}, x_{2eq}, \hat{u})}^{\,0} + \left.\frac{\partial f_1}{\partial x_1}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta x_2 + \left.\frac{\partial f_1}{\partial u}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta u$

$$\Delta\dot{x}_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta x_2 + \left.\frac{\partial f_1}{\partial u}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta u$$

Same thing for $\Delta\dot{x}_2$. Rewriting in matrix form gives

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{x_{1eq}, x_{2eq}, \hat{u}} \Delta u \quad \Rightarrow \quad \Delta\dot{\underline{x}} = J\,\underline{\Delta x} + \underline{b}\,\Delta u$$

J is the Jacobian matrix.

<u>Ex</u> $\frac{dy}{dt} = y(1-z) \qquad \frac{dz}{dt} = z(y-1)$

<u>Equilibrium:</u> $y=0, z=0;\ y=1, z=1$ $\qquad$ Linearize around $y=1, z=1$

$$J = \left.\begin{bmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}\right|_{y=1, z=1} \Rightarrow J = \left.\begin{bmatrix} 1-z & -y \\ z & y-1 \end{bmatrix}\right|_{y=1, z=1} \Rightarrow J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Delta y \\ \Delta z \end{bmatrix}}$$

Linearization-3

Ex $m\ddot{x} = -k(x-\ell)\left[1+\left(\frac{(x-\ell)}{\ell}\right)^2\right]$

In state space:

$$\frac{dx}{dt} = \dot{x} \qquad\qquad x_{eq} = \ell$$

$$\frac{d\dot{x}}{dt} = -\frac{k}{m}(x-\ell)\left[1+\left(\frac{x-\ell}{\ell}\right)^2\right] \qquad\qquad \dot{x}_{eq} = 0$$

$$J = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \dot{x}} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \dot{x}} \end{bmatrix}\right|_{x=\ell,\dot{x}=0} \Rightarrow J = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta \dot{x} \end{bmatrix}}$$

Ex $m\ddot{r} = -k(r-R_0) + mg\cos(\theta) + mr\dot{\theta}^2$

$mr\ddot{\theta} = -mg\sin(\theta) - 2m\dot{r}\dot{\theta}$

Equilibrium point: $r = R_0 + \frac{mg}{k}$ $\quad \theta = 0$ $\quad , \dot{r} = 0, \dot{\theta} = 0$

State Space:

$$\frac{dr}{dt} = \dot{r}$$

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\frac{d\dot{r}}{dt} = -\frac{k}{m}(r-R_0) + g\cos(\theta) + r\dot{\theta}^2$$

$$\frac{d\dot{\theta}}{dt} = -\frac{g}{r}\sin(\theta) - 2\frac{\dot{r}\dot{\theta}}{r}$$

$$J = \left.\begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \dot{r}} & \frac{\partial f_1}{\partial \dot{\theta}} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \dot{r}} & \frac{\partial f_2}{\partial \dot{\theta}} \\ \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \dot{r}} & \frac{\partial f_3}{\partial \dot{\theta}} \\ \frac{\partial f_4}{\partial r} & \frac{\partial f_4}{\partial \theta} & \frac{\partial f_4}{\partial \dot{r}} & \frac{\partial f_4}{\partial \dot{\theta}} \end{bmatrix}\right|_{\text{equilibrium}} \Rightarrow \boxed{J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m} & 0 & 0 & 0 \\ 0 & -\frac{g}{R_0+\frac{mg}{k}} & 0 & 0 \end{bmatrix}}$$